First Symposium on Artificial Intelligence for Mathematics Education

Philippe R. Richard
Steven Van Vaerenbergh
M. Pilar Vélez
(editors)

Book of Abstracts

February 28th - March 1st, 2020
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First Symposium on Artificial Intelligence for Mathematics Education.

Book of Abstracts

(AI4ME 2020)

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First Symposium on Artificial Intelligence for Mathematics Education. Book of Abstracts (AI4ME 2020)

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Hecho en España-*Made in Spain*

Santander, 2020

Introduction

The digital revolution that we have experienced since the last quarter of the twentieth century, with the arrival of pocket calculators, computers, tablets, smartphones, etc. has had some influence, yet to be analysed and extended, on the way mathematics is made, taught and learned. Clearly, the potential impact of most information technologies on mathematical education has not yet been fully exploited, while the rate of innovation in these technologies is growing exponentially.

In particular, several authoritative voices point out that the technology that will most likely transform education in the coming years is artificial intelligence (AI). Interestingly, today AI is mainly associated with technologies to automate tasks and lower costs, thus serving primarily the interests of the political-administrative, industrial and commercial world. In this scenario, the world of education and, more specifically, didactics, appears at best as a mere user of AI techniques developed in other fields, forgetting that AI should play a much more relevant role here, serving the human being who is doing his work as a mathematician or who is learning mathematics.

The Symposium on Artificial Intelligence for Mathematics Education (AI4ME 2020)[1] was held at the CIEM (International Centre for Mathematical Meetings) in Castro Urdiales, in the Spanish region of Cantabria. The AI4ME Symposium has been a space for research and reflection to better understand the interconnected challenges of instrumental learning of mathematics and instrumental mathematics, taking advantage of the achievements and opportunities of Artificial Intelligence for Mathematical Education. The Symposium was developed through individual presentations by one or more invited experts contributing to working and discussion groups around different topics that collect different points of view, and the sharing of conclusions.

This book of abstracts gathers the summaries of the talks, as well as the conclusions of each of the thematic groups, namely:

1. *STEM and classroom experiences*, presided by Belén Palop (Universidad de Valladolid, Spain);
2. *Digital tools for mathematics education and instrumental reasoning*, presided by Jana Trgalová (Université Claude Bernard Lyon 1, France);
3. *Dynamic geometry for mathematics education*, presided by Eugenio Roanes-Lozano (Universidad Complutense de Madrid, Spain);
4. *Virtual reality, artificial intelligence and machine learning for mathematics education*, presided by Theodosia Prodromou (University of New Eng-

[1] `https://ai4me.unican.es/`

land, Australia) and Steven Van Vaerenbergh (Universidad de Cantabria, Spain).

The organization of the Symposium has been supported by CIEM (International Centre for Mathematical Meetings, ICMM) in Castro Urdiales, Universidad de Cantabria and Fundación Caja Cantabria; and sponsored by Universidad Nebrija, Laboratoire Turing of the Université de Montréal, Hotel Las Rocas and Ayuntamiento de Castro Urdiales.

All this has been possible thanks to the involvement of all participants and the scientific committee. We thank everyone for their contribution, for the great atmosphere of collaboration and the interesting discussions. A special mention goes to Tomás Recio (the "shadow co-organizer") who encouraged us to propose this symposium and advised us in the previous tasks.

The city of Castro Urdiales provided the "real part" —not "artificial", nor "virtual"— welcoming us with a spring weather in February that allowed us to enjoy its walks, its landscapes and, of course, its gastronomy.

Finally, we would like to point out that this event took place at the very dawn of the global pandemic. Since fortune favours the bold, we hope to see you in Castro Urdiales for the next edition of AI4ME!

September 2020

The AI4ME organizers:
Philippe R. Richard
Steven Van Vaerenbergh
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Table of Contents

Topic Group 1. STEM and Classroom Experiences

Topic Group 2. Digital Tools for Mathematics Education and Instrumental Reasoning

Topic Group 3. Dynamic Geometry for Mathematics Education

Topic Group 4. Virtual Reality, Artificial Intelligence and Machine Learning for Mathematics Education

TG1 Panel: STEM and Classroom Experiences

Belén Palop del Río

Universidad de Valladolid, Spain

This is a summary of the *TG1 Panel: STEM and Classroom Experiences* that took place at the Symposium on Artificial Intelligence for Mathematics Education (AI4ME), held at CIEM Castro Urdiales, February 28th - March 1st, 2020. Full list of authors: Belén Palop, Zsolt Lavicza, Elena Alvarez, Jean-Baptiste Lagrange, José Manuel Diego Mantecón, and Zaira Ortiz.

Summary

The panel started with the invited talk by Zsolt Lavicza, *STEAM education approaches and technological innovations to foster creativities in schools in a digital era*, and was followed by four short presentations:

1. Jean-Baptiste Lagrange, *Connected working spaces: modelling in the digital age*
2. Elena Alvarez, *Teachers' perspective on some STEAM/AI oriented European Projects*
3. Zaira Ortiz Laso, *STEAM activities with KIKS format*
4. Belén Palop, *STE(A)M Lessons Learned: from STEM4Math to STEAM-CT*

Since the summary of each talk can be seen in the abstracts, let us point out here the main ideas that were shared by the speakers.

What is STE(A)M Education?

Zsolt Lavicza and Belén Palop focused on the different levels of integration of the subjects when we talk about STEAM Education. In the lowest level, we have the most simple connections that are maybe just mentioned by the teacher. Higher up in the integration level, we need to move in the direction of the meaningful and deep ways of thinking in the intersection of several subject like, for example the connections between Physics and Architecture with mathematical modeling explained by Jean-Baptiste Lagrange. Unfortunately, as Belén pointed out, educators usually have no personal experience as learners in a STEAM setting and not many of them have a profound knowledge on how to bring STEAM Education to their classrooms.

Transdisciplinary levels, where all subjects are intertwined are more difficult to achieve and need further research, as performed in the PhD program lead by Zsolt Lavicza, where the essence of STEAM Education and its connections is been investigated.

Who does STEAM Education?

As presented by Elena Alvarez, teachers still need more help from the automated systems to guarantee that AI is helping them be more efficient. Unfortunately, their perception is still that they need to learn more, work harder or be more motivated to be able to use these tools in the classroom. The new challenges involve thus to provide them with more finished tools that can adapt to the permanent changes in the underlying technology, the small changes in the curriculum, or the lack of financial support, among others.

Not only teachers can implement STEAM Education, as shown by Zaira Ortiz Laso. In its own core, STEAM Education is connected to active-learning, group-work and problem solving. Zaira's team presented their experience allowing children do peer teaching and learning. Children's engagement in their learning process is boosted through the elaboration of their own online learning materials.

I'd like to thank Pilar Vélez, Steven Van Vaerenbergh and Philippe R. Richard for letting me chair this panel and for the fruitful discussions during the whole symposium.

STEAM education approaches and technological innovations to foster creativities in schools in a digital era

Zsolt Lavicza

Johannes Kepler University, Austria

Besides tackling challenges and disruptions caused by digital technologies in schools, there is also a growing emphasis for encouraging creative thinking in education, innovating pedagogies and develop connections among subjects. Activities focusing on creative processes, rather than concentrating on achieving only results for posed problems, are being designed and trialled by innovative groups around the world. In my talk, I will introduce ideas and examples for technological, pedagogical and policy innovations involving STEM to STE-A-M (by the inclusion of Arts in a broader sense of creation and creativities) transitions. These examples will include STEAM research with the Experience Workshop Movement; studies related to GeoGebra and it new developments such as Augmented Reality, 3D Printing, Machine Learning and Mobile experiments; developing students' skills through robotics and connecting digital and physical worlds; and possibilities to detect and nurture creative thinking processes from Big Data. An overview of such studies could offer new insights into developments of creativities, innovations for teaching and learning, and opportunities for nurturing further collaboration in these areas.

Connected working spaces: modelling in the digital age

Jean-Baptiste Lagrange

LDAR University of Paris, France

In mathematics education, many researchers consider modelling as a "translation between reality and mathematics" restricting the potential of modelling activities with regard to the diversity of approaches to a given reality, especially those allowed by technology, and the rich connections they can help students build between various fields of knowledge.

From epistemological studies, I stress that modelling is not merely mathematizing: for a given reality, there is a plurality of approaches and models, and mathematical work is done in close conjunction with work in other scientific experimental fields.

Considering a plurality of models of a given reality, each pertaining to a specific scientific field, I take advantage of the notion of working space in order to make sense of modelling activities. I give the example of 12th grade students modelling a suspension bridge. Students work on four models and make connections between the associated working spaces: a physical model (mock-up) with instruments and rules of physics, a geometrical model explored in paper/pencil, a dynamic simulation involving computer programming and a dynamic functional model with the help of computer algebra.

Teachers' perspective on some STEAM/AI oriented European Projects

Elena Esperanza Alvarez Saiz

Universidad de Cantabria, Spain

In our presentation we will, first, shortly introduce two Erasmus+ projects: the project "LEARN+/MILAGE: MathematIcs bLended Augmented GamE" and the project "MoMaTrE (Mobile Mathematics Trails in Europe)" and its continuation "MaSCE^3 (Math trails in School, Curriculum and Educational Environments in Europe)", focusing on some of its features that could be closer to AI: gamification, augmented reality, smartphones… for mathematics learning.

Then we will reflect, from the point of view of teachers: more precisely, from the point of view of the large collective of teachers conforming the Spanish Federation of Math Teachers Societies, on how these innovative projects and tools could be actually considered in the standard, daily classroom context. We will describe and evaluate our experiences and on-going work as members of the consortium for the projects.

STEAM activities with KIKS format*

Zaira Ortiz-Laso

Universidad de Cantabria, Spain

Most European countries have adapted their curricula towards the acquisition of key competences in education [1]. This implies students applying school knowledge to solve a problematic situation, which often requires the application of knowledge from different areas. Consequently, several educational systems have employed the STEAM (Science, Technology, Engineering, Arts, and Mathematics) education approach to develop student key competences, through different national and international projects. In Spain, the Open STEAM Group has promoted different initiatives to encourage middle and high school students in the learning of the STEAM subjects, and particularly in the learning of mathematics[1].

The Open STEAM Group initiatives are normally implemented through the application of the STEAM project-based learning methodology with KIKS format. The project based-learning methodology implies the development of high-tech projects under three fundamental pillars: interdisciplinarity, inquiry-based learning, and collaborative work [2]. KIKS is the acronym of Kids Inspire Kids for STEAM [3]. The KIKS format implies the development of two files per project, collecting information about their practical and analytical parts, as well as the dissemination of the project in events. The Open STEAM Group has several online platforms where projects are offered to teachers and students. These projects are designed by experts under an adaptation of Thibaut et al.'s framework [4], and students have to develop them in a non-maternal language, usually English. Students are asked to work collaboratively, in groups of 3-4 members, and are supervised by several teachers. Each group develops at least two projects over a period of at least two years. When the project is completed, students produce in their non-maternal language a report, including the project description, its development, and its results, as well as a video, containing explanations about its applicability in real life. In addition, students present their project in face-to-face events and videoconference to a variety of audiences. The events in which students exhibit their STEAM projects are devoted, for example, to teachers, students, and researchers.

* Work carried out within the frameworks of the projects: STEMforYouth (European Union's Horizon 2020 Programme, under grant agreement 710577), EduMath (European Union's Erasmus+ Programme, under grant agreement 2019-1-CZ01-KA201-061377), and EAMARE-STEAM (FEDER/Ministerio de Ciencia, Innovación y Universidades – Agencia Estatal de Investigación/ under grant agreement EDU2017-84979-R).

[1] `https://www.opensteamgroup.unican.es/`

The Open STEAM Group often frames the STEAM based-learning projects with KIKS format under different national and international initiatives, according to the current trends and necessities of the society. For example, the STEMforYouth project[2] [5] (Horizon 2020) is aimed at secondary school students who follow the regular curriculum, while the EAMARE-STEAM project[3] [5] (Spanish Ministry of Education) seeks to motivate secondary school students at risk of exclusion. In evaluations of both projects it has been observed, though at different levels, a significant improvement in the development of the main key competences highlighted by the European Union [3]. The implementation of this project has been also effective to generate students' positive beliefs about STEAM disciplines and, particularly, about mathematics [6].

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[2] `https://stemforyouth.unican.es/`

[3] `https://www.inclusivemathsthroughsteam.unican.es/`

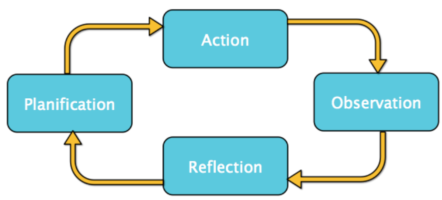
STE(A)M Lessons Learned: from STEM4Math to STEAM-CT

Belén Palop del Río

Universidad de Valladolid, Spain

In 2016, our group at the University of Valladolid received an invitation to participate in an Erasmus project to exchange innovative ideas and practices to improve Math learning through STEM Education. More than 150 Primary School children and their teachers in Spain and dozens of classrooms in the other four participant countries (Finland, Sweden, Belgium and Portugal) tested our approaches to the topic and helped us improve 20 project ideas[1].

All projects were developed following an instructional design schema, where school teachers were coached by the project teams and improvements were made along the process.



This project successfully ended in the summer of 2019, when we were granted a follow-up on the topic. The project called STEAM-CT focuses on Computational Thinking and adds the countries of Lithuania and Italy to the strategic partnership. Moreover, it opens-up our focus to smaller children (4/5 year olds) and progresses up to Secondary School students.

The main outcomes of STEM4Math will be presented in this talk together with the lessons that we have learned during the process.

[1] http://stem4math.eu

Classroom Implementation of STEM Education through technology: advantages and handicaps*

Jose Manuel Diego-Mantecón

Universidad de Cantabria, Spain

Open STEAM Group investigations[1] in the last five years suggest that the classroom implementation of the STEM approach with a high technological component has advantages and handicaps [1]. 2056 middle and high school students from six countries—Poland, Italy, Greece, Slovenia, Czech Republic, and Spain—elaborated STEM activities over two years, as part of the STEMforYouth project (Horizon 2020). The students from 68 educational centers worked in groups of 3-4 members supervised by at least one teacher; involving a total of 120 instructors [2]. The analyses revealed that the STEM approach—with specific emphasis on the technological component—developed positive beliefs towards the usefulness of the STEM disciplines and increased learning motivation [3]. The STEM activities, designed under the project-based learning methodology with KIKS format, turned out also effective to develop the key competences established by the European Union [1].

The analyses suggested, however, that the implementation of certain STEM activities that require several sessions to be completed do not fit well within the schedule of the regular curricula, nor within the evaluation methods of the current educational systems. Another significant drawback of the STEM approach implementation was the lack of teachers' knowledge in the interdisciplinary component [4]. The majority of teachers are subject-specific; this generates a low self-confidence and anxiety when professionals have to deal with interdisciplinary activities [3]. The fact that there is no continuous training also produces teacher's insecurities when using new technologies [1]. In some contexts, the low communication and collaboration between teachers from different areas do not benefit the STEM approach implementation. The absence of colleagues' support (e.g. teachers from the same school and management staff) is also a significant barrier for not adequately implementing STEM activities [5, 6]. Although, to a lesser extent, the lack of resources (e.g. materials and laboratories) also makes it difficult to develop certain STEM activities in the educational centers. The ongoing

* Work carried out within the frameworks of the projects: STEMforYouth (European Union's Horizon 2020 Programme, under grant agreement 710577), Edu-Math (European Union's Erasmus+ Programme, under grant agreement 2019-1-CZ01-KA201-061377), and EAMARE-STEAM (FEDER/Ministerio de Ciencia, Innovación y Universidades – Agencia Estatal de Investigación/ under grant agreement EDU2017-84979-R).

[1] `https://www.opensteamgroup.unican.es/`

investigations carried out within the framework of the EAMARE-STEAM project reinforce the conclusions described above [7].

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TG2 Panel: Digital Tools for Mathematic Education and Instrumental Reasoning

Jana Trgalová

Université Claude Bernard Lyon 1, France

This is a summary of the *TG2 Panel: Digital Tools for Mathematic Education and Instrumental Reasoning* that took place at the Symposium on Artificial Intelligence for Mathematics Education (AI4ME), held at CIEM Castro Urdiales, February 28th - March 1st, 2020.

Summary

The TG2 Panel focused on digital technologies supporting learners' mathematical activity.

The session was introduced with a plenary talk by Jana Trgalová "Digital technology and its various uses from the instrumental perspective". On the example of dynamic geometry, she showed that mathematical tasks involving a digital tool can benefit to lesser or greater extent from its affordances, from a mere substitution of "traditional" non-digital tools to tasks that cannot exist but within the digital tool.

The three following interventions illustrated various roles digital tools can play in mathematics education and thus support mathematics learning:

1. Eunice Chan and Robert Corless in their contribution *Teaching programming to mathematics scientists* shared their experience with a blended course on computational mathematics. Programming visually appealing Newton fractals and other playful activities with Maple or Python not only helped students overcome difficulties frequently encountered when learning programming, but also triggered better conceptualization of mathematical notions at stake.
2. The contribution *Understanding and creating to better understand instrumental proof using QED-Tutrix* by Philippe R. Richard presented new developments of a system specifically designed to supports students in solving problems of proof. The system embeds a virtual pedagogical agent capable of following students solving a proof problem, which is based on the referential of school mathematics properties and definitions. The new developments consist in considering, besides verbal justifications of inferences, a wider range of justifications, such as justifications provided by a technological tool, the construction of a dynamic figure, or the execution of an algorithm.

3. Tomás Recio in his contribution *Towards a mechanical geometer* reported about a development of a system aiming at automated discovery of properties in elementary geometry recently implemented to Geogebra. The capability of the system to discover and prove geometric properties raises important didactic issues about the role of digital tools in mathematics teaching and learning or the impact of digital tools on mathematics curricula.

Discussions triggered by the presentations brought to the fore several important issues related to the theme and led to questioning the very link between artificial intelligence and mathematics education.

First, when digital tools are referred to in relation with mathematics education, one naturally thinks about tools supporting students' learning of mathematics, such as Geogebra, Maple or QED-Tutrix, to name only those that were mentioned in the presentations. These tools support students' mathematical activity, and consequently their learning, by providing mathematical or didactic feedback [1]. Whereas mathematical feedback aims at helping students make sense of the phenomena observed on the interface (e.g., invariance of geometric properties while dragging free points in Geogebra), the role of didactic feedback can be to evaluate students' responses (true or false) or to support them in the task resolution (e.g., scaffolding as in QED-Tutrix). The latter usually requires deeper didactic analysis of the mathematical domain at stake and of the possible students' reasoning strategies in order to provide relevant feedback in response to students' actions [2]. On the other hand, digital tools providing teachers with information about their students to help their decision making are scarce [3, 4]. The development of such tools benefits from artificial intelligence methods to model students' (mis)conceptions and pedagogical strategies and to compute adequate didactic responses to students' actions.

Another important issue that was raised during the discussions concerned the potential of digital tools. Some tools offer the possibility to provide a-didactic milieu [5], with which the students interact and get (mathematical) feedback that they need to interpret (e.g., Geogebra). Other tools create a didactic milieu with explicit (didactic) feedback, for example about the validity of the provided response or suggesting next step in the problem solving (e.g., QED-Tutrix). Yet some other tools amplify the user capabilities by providing answers (e.g., Maple) or performing tasks (e.g., Geogebra Automated Geometer). The availability of such tools raises a number of questions, in particular:

1. Do the students need to learn how to solve tasks the tool can solve? This question addresses the issue of the impact of the use of digital tools on mathematics curriculum.
2. How can such potential of digital tools be exploited for purposes of mathematics teaching and learning? This question opens avenues toward designing new types of tasks.

Regarding the instrumented learning, the discussions brought to the light the importance of considering the semiotic potential of the digital tool [6] in order to be aware which mathematical meanings it conveys. Instrumental issues need also to be taken into account: indeed, while using a tool to accomplish a given task, a user develops a personal instrument [7] that can differ from one user to the other, depending on their knowledge or beliefs. These considerations lead to rethink the role of the tutor, whether human, virtual or blended, which may change in a digital environment, but remains crucial in accompanying the students toward the achievement of the target educational goal.

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Digital technology and its various uses from the instrumental perspective

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In 1985, Pea contrasted the use of computers in an amplification and reorganization metaphors: in the former, technology allows performing tasks faster, more efficiently and accurately, whereas in the latter, technology qualitatively changes “both the content and flow of the cognitive processes engaged in human problem solving” [1, p. 170]. In this talk, we take dynamic geometry as an example of digital technology to illustrate various ways in which it can be used, referring to the SAMR model [2]. Drawing on the instrumental approach [3], for each kind of use, we analyze the role of the drag mode to highlight a variety of instruments that can be developed and the corresponding conceptualizations. We conclude with some implications bringing to light challenges that mathematics teachers face with the use of technologies.

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Teaching Programming to Mathematical Scientists

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We recount our experience teaching an experimental mathematics course using computer algebra and other computational tools. We used active learning methods [1, 2]. The central theme of this work is that new tools for mathematics, such as are embodied in computer algebra, require changes in the *content* of our courses, and do not merely allow the same content to be presented in a different way [3–5].

This particular course is described in greater detail in [6, 7]. Through this "experimental" course, we intended that students' mathematical abilities would be strengthened by learning programming that was focused on mathematics. We tried to avoid the standard curriculum, in order to keep student interest high. We used continued fractions [8], fractals [9–11], Bohemian Matrices[1] [12, 13] and iterated function systems and the chaos game representation [14].

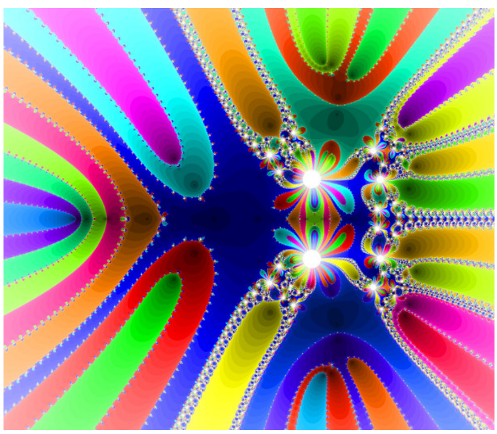

Fig. 1. Newton fractal of a Mandelbrot polynomial (generated using Python).

The main purpose of the course was to get students more comfortable with experimentation and computational discovery. To do this, we helped them to build their own tools (and gave them access to existing tools, using Maple, Matlab, and Python). This approach is not new: for a description of

[1] http://www.bohemianmatrices.com/

how this can be done in a purely numerical computation environment, see [15]. Even in a more algebraic or geometrically oriented course, however, one cannot avoid profound implications of floating-point arithmetic.

> "Admit, for instance, the existence of a minimum magnitude, and you will find that the minimum which you have introduced, small as it is, causes the greatest truths of mathematics to totter." [16].

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Understanding and creating to better understand instrumental proof using QED-Tutrix

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The understanding and modelling of the conditions for learning mathematics, together with the creation of models and computational means to understand them, are at the heart of an emerging research. The QED-Tutrix system, from an earlier project that supports the student in solving problems of proof, has been designed with respect to the discourse habits of the classroom and has been developed with a focus on the designer/user dialogue. In this system, the original creation of inferential graphs, associating a set of structured reasoning to the statement of a problem so that a virtual pedagogical agent can follow the student in his proof, is based on the referential of mathematical properties and definitions used in schools. Thus, the justification of a reasoning step is made according to this referential and allows legitimizing the necessity in the linking of knowledge. However, until now, these justifications have been strictly verbal, following the example of reasoning in traditional mathematics. What happens if some inferences are justified by an interacting technological tool, such as calculations (numerical or symbolic), the construction or animation of a dynamic figure, the execution of an algorithm, the creation of a recognized mathematical process, the use of an automated reasoning tool or the modelling of a real-life situation?

Towards a mechanical geometer*

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Our presentation deals with the mathematical, technological and social issues involved in the current development of a *mechanical geometer*, built on top of GeoGebra, a free, dynamic mathematics system, available on and offline, in different devices (computer, laptops, tablets, smarthphones), with more than 100 million users all over the world[1].

The idea of a *mechanical geometer* is a legendary goal (consider, for instance, the work of Leibniz in the XVIIth century, or the *Plane Geometry* book *from the future* by Doron Zeilberger [1]). The actually performing *mechanical geometer* that we will present in our communication receives, as input, a geometric figure drawn by the user using the different tools of GeoGebra. Then it interprets this figure as a generic representative of a collection of geometric constraints holding among the different elements of the construction.

Finally, this *mechanical geometer* offers the user a variety of possible geometric tasks: automatically discovering the relation holding among different elements of the figure, automatically discovering "all statements" holding true among points in the figure selected by the user, automatically proving the truth or failure of a given statement, or automatically discovering how to modify a given figure so that a wrong statement becomes true. Let us remark that some of these tasks, are, as far as we know, quite unique in the context of automated reasoning, and they originated in [2]. See [3] for more details on the performance of this tool.

Roughly speaking, the mathematics behind the *mechanical geometer* involve the translation of the geometric facts into a collection of polynomial equations and inequations, and the corresponding manipulation by means of (Complex or Real) computational algebraic geometry algorithms developed by the authors, involving Hilbert dimension, ideal elimination and saturation computations using the free computer algebra software Giac [4] embedded in GeoGebra for Gröbner Bases computations and, also, some freely available tools for Cylindrical Algebraic Decomposition – in the real case [5].

Moreover, we will address the potential impact of this *mechanical geometer* in different contexts, such as the educational [6] or research [7] world.

The possible contribution of this device to develop some *automatically augmented reality* app, starting with the automatic transformation (e.g. by means

* This is work in collaboration with F. Botana, Z. Kovács and M.P. Vélez. We all are members of the FEDER/Ministerio de Ciencia, Innovación y Universidades - Agencia Estatal de Investigación/MTM2017-88796-P (Symbolic Computation: new challenges in Algebra and Geometry and their applications) research project

1 `http://www.geogebra.org`

of the Hough transform) of the image of a real object into a geometric figure in GeoGebra and then allowing the automatic analysis of its geometric characteristics, will be also highlighted, see [8]; or [9]. Obviously, the *mechanical geometer* could also play a relevant role in the configuration of a digital, intelligent book of geometry [10].

In our presentation all these issues: some of them already accomplished, already real; some of them yet just conceived, imaginary … complex … will be summarily described, exemplified and discussed.

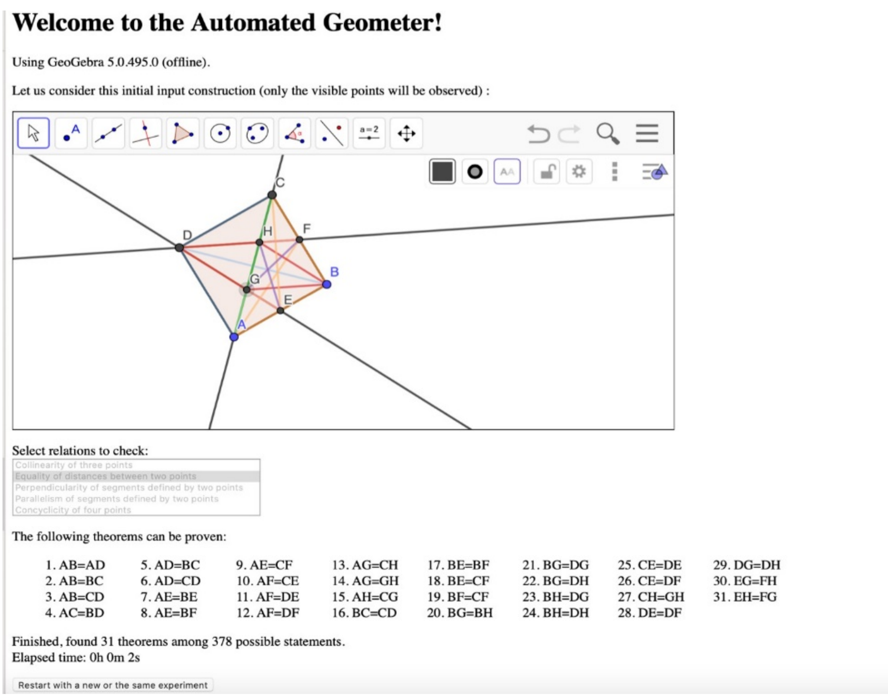

Fig. 1. A square ABCD is built on the Automated Geometer window, with lines from vertex D to the midpoints of the opposite sides and diagonal AC, as well as the corresponding intersection points. The user asks the Automated Geometer to find statements concerning equality of distances between points, that hold universally under such constraints. In 2 seconds, 31 theorems are found.

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TG3 Panel: Dynamic Geometry for Mathematics Education

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This is a summary of the *TG3 Panel: DGS for mathematics education* that took place at the Symposium on Artificial Intelligence for Mathematics Education (AI4ME), held at CIEM Castro Urdiales, February 28th - March 1st, 2020.

Summary

The *TG3 Panel: Dynamic Geometry for Mathematics Education* focuses on the influence of the arrival of Dynamic Geometry Systems (DGS) on mathematics education. Nowadays, the evolution of the possibilities of the available DGS and the unprecedented spread of the DGS GeoGebra lead to many open questions, like:

– Do teachers (and textbooks) really take advantage of all the possibilities of new DGS?
– Should DGS affect to the way we teach? (for instance, should we teach in a more experimental way?)
– Should DGS change what we teach?
– Should DGS change how we structure what is taught?

One excellent invited talk plus four exciting talks addressing applications of DGS to mathematics education took place in this panel:

– Pedro Quaresma: *Proofs & Dynamics in Geometry.*

1. Pilar Vélez: *A short introduction to GeoGebra automated reasoning tools (ART)*.
2. Thierry Dana-Picard: *Automated exploration of envelopes*.
3. Cristina Naya: *Teaching of Geometry with GeoGebra software in students of the Primary Education Degree*.
4. Eugenio Roanes-Lozano: *A constructive educational approach to conics and quadrics allowed by the arrival of 3D-DGS*.

that was followed by a discussion and final conclusions.

In my opinion, the invited talk and the four talks look very different, but there is a common core: the important influence of DGS in teaching mathematics and the collateral pedagogical issues that arise.

There are two eye catching issues in the present DGS development:

– The extension of 3D capabilities.

– The collaboration with Computer Algebra Systems (CAS), that opens a new world of possibilities if compared with "standard" DGS, and can be applied to all levels of mathematics education, from Secondary Education to Teacher Training and first years at universities (both Science and Engineering schools).

Let us continue with a summary of the talks and the ulterior discussions.

Pedro Quaresma described in his invited talk how dynamic geometry systems (DGS) provide large sets of cases (through dynamism) that give a clue to the truthfulness of geometric results, meanwhile geometry automated theorem provers (GATP) go beyond and produce formal proofs. He underlined how both DGS and GATP enhance learning in different ways and the relevance of immediacy and readability in the educational applications of the latter.

Pilar Vélez gave a summary of the possibilities of an impressive new tool (still under development) for automated reasoning in GeoGebra. This extension of GeoGebra opens a new field in maths teaching, as, for the first time since he beginning of mathematics teaching with technology, the students can not only explore but also obtain a confirmation of the formal truthness of a geometric result (and even been suggested new results).

Thierry Dana-Picard showed us a surprising example about how "dragging and adjusting" with a DGS could make possible for Secondary Education students to face tasks (in this case determining the envelope of a family of curves) that, without technology would be impossible to achieve at this level.

Cristina Naya gave examples of how the (rule and compass) inconstructability and the inexistence of certain geometric configurations could be treated with a DGS at Teacher Training level. Curiously, she underlined the rejection of technology by some of the students, although we (teachers) consider that it is engaging for all young people.

Finally, my talk presented how a 3D DGS with algebraic capabilities could be used to introduce conics as sections of a right circular (Apollonius' cone) in a constructive way (not so algebraic and more visual). It was shown how quadrics of revolution can be also presented in a constructive way. A question arose at the end: whether these topics should be taught or not or to whom.

Some didactics ideas arose along the final discussion:

– Should geometry curricula change due to the availability of the new DGS?
– Thinking about an average Secondary School student, the importance of formal proofs possibly decreases with tools that allow to convince through checking the "stability" of a construction (and/or the answer of a black box for automatic theorem proving).
– These powerful DGS allow to explore at Secondary School level issues allocated at university level (that is, to "lower" contents).

- DGS didactic possibilities are usually compared to those of rule and compass, but they are key to treat questions such as inexistence or inconstructability of a geometric configuration.
- Similarly, the approach of the last talk requires a (very special) DGS.
- Finally, the didactical possibilities of 3D DGS with virtual reality look huge.

To conclude, DGS are evolving very quickly lately, specially GeoGebra, after some years of a "stabilization" of their goals (let us remember that the first DGS appeared back in the early 90s). They are becoming THE TOOL for geometry teaching and are opening new possibilities and ways to teach.

I would like to thank the organizers of AI4ME, Philippe R. Richard, Steven Van Vaerenbergh and Pilar Vélez for inviting me to chair this panel.

Proofs & Dynamics in Geometry

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Given its formal, logical, and spatial properties, geometry is well suited to teaching environments that include dynamic geometry systems (DGSs), geometry automated theorem provers (GATPs), and repositories of geometric problems. These tools enable students to explore existing knowledge in addition to creating new constructions and testing new conjectures.

Dynamic geometry programs give users an initial visual validation of a geometric property. Instead of producing a fixed example, these programs produce a large set of examples that reinforce confidence in the truth of a statement. Although these manipulations are not formal proofs because only a finite set of positions is considered and because visualisation can be misleading, they provide a first clue to the truthfulness of a given geometric conjecture. It can be said that DGSs provide an initial non-formal link between theories and models of geometry.

Geometry automated theorem provers similarly enhance learning. They can be used to validate a given conjecture about a geometric construction or, better, to produce a formal proof of it.

Efficiency is important because, in a learning situation, it is not viable to wait more then a couple of seconds to get an answer. This opens the door to GATPs implementing algebraic methods and also the discussion about taxonomies and measures of complexity for proofs.

Readability is important because, without it, the “why” would be lost. This opens the door to GATPs implementing synthetic and semi-synthetic methods and also to the rendering of proofs in natural languages and, important in geometry, visual languages.

A short introduction to GeoGebra automated reasoning tools (ART)*

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Recently, the computer algebra system *Giac* was embedded in *GeoGebra* [1], allowing for the implementation of automated proving algorithms based on the algebraic approach described in [2].

The result is a collection of GeoGebra tools and commands that allow to conjecture, discover and prove statements on a given geometric construction: ART features are available since GeoGebra 5 and new ART improvements and features can be found in GeoGebra-Discovery Beta version[1].

This communication attempts to introduce, describe and exemplify the technical features of some recently implemented Automated Reasoning Tools in the dynamic mathematics software GeoGebra. As well as to discuss on the benefits and concerns arising from the use of these automated tools in the mathematical teaching and learning process.

Automated Reasoning Tools (ART) available in *GeoGebra* include several commands (a complete tutorial can be found in [3]): the `Relation` command, that can be used for the automatic finding of geometric conjectures and the verification or denial of these conjectures; the `LocusEquation` command, which calculates the implicit equation of a free point such that a given property holds; the `Prove` and `ProveDetails` commands, which decide if a statement is true in general and, eventually, give some additional conditions for its truth, avoiding degenerate cases; and the `Envelope` command that computes the equation of a curve which is tangent to a family of objects while a certain parent of the family moves on a path. The *GeoGebra-Discovery Beta version* includes improvements of these commands and a new feature, the `Discover` command which gives statements holding true involving one element selected by the user in the figure.

Our aim is that the use ART in the classroom help students to do mathematics better or faster, in a more creative way by exploring, discovery and conjecturing, which fosters their curiosity and critical spirit, as well as gives them a way of reasoning focused on competencies for the digital age. The challenge is to move from automated reasoning tools for math learning (use artefacts to achieve some didactic goals) to Instrumented learning for math reason-

* This is work in collaboration with F. Botana, Z. Kovács and T. Recio. We all are members of the FEDER/Ministerio de Ciencia, Innovación y Universidades - Agencia Estatal de Investigación/MTM2017-88796-P (Symbolic Computation: new challenges in Algebra and Geometry and their applications) research project.

[1] `http://autgeo.online/geogebra-discovery/`

ing (reconsider didactic goals of math education according to new computer tools).

We hereby state that this challenge requires a new design of tasks, more open to conjecture, investigation and verification (see Fig. 1 and 2). For instance, in the frame of the The SINUS Project 1998-2007, open-ended tasks are considered in a more general context. *"Open-ended tasks are any tasks where students are asked to explore objects and to discover and investigate their mathematical properties"* [4, p. 23]. Some examples of tasks based in ART can be found in [5].

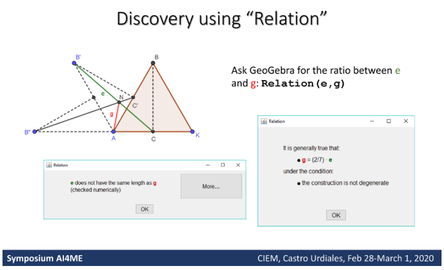

Fig. 1. ABK is an equilateral triangle, B′ and B″ are the symmetric of K about C′ and A resp. Ask to discover the relation between segments e (green) and g (red).

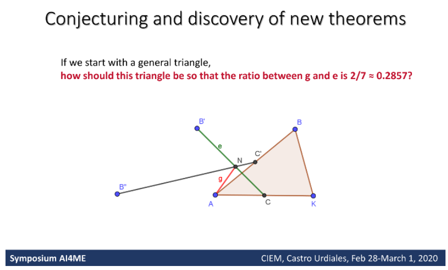

Fig. 2. Investigate if there are more triangles than the equilaterals, fulfilling the 2/7 relationship between e and g in a similar construction.

While the classroom use of ART is still in an incipient and experimental phase, one should bear in mind that these tools are now readily available to the more than 100 million *GeoGebra* users worldwide. Hohenwarter, Kovács and Recio [6] note that, *"as with pocket calculators, people will probably start using ART for checking geometric facts without the consensus of the pedagogical community on its role"*.

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Automated exploration of envelopes

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The study of envelopes is a classical topic. One of its characteristics is the small number of theorems, therefore the study of envelopes of families of plane curves and of surfaces in 3D space is mostly made of the study of special cases.

Tools for automated study of these objects have been developed during the last years, in Dynamical Geometry Systems (DGS), for example in GeoGebra. A joint usage of the dynamical features (slider bar, mouse dragging) and of implemented commands (**Envelope**(`<Path>,<Point>`)) may yields double answers: graphical – a plot of the desired envelope, and algebraic – an implicit equation for the envelope. A graphical answer may provide a conviction that an envelope exists, without being a full proof. Of course, the family of curves (resp. surfaces has to be defined in a format suitable for the command to be effective).

Nevertheless, this kind of double answer is not always available, for programming reasons. For example, GeoGebra's **Envelope** command requires a geometric construction of the dependence between mover and tracer. It cannot be used when the family is defined analytically (such a case is studied in a subsequent work). Another kind of software, namely a Computer Algebra System (CAS), may be useful to compute a parametric presentation of the desired envelope (a result who is a proof of the existence of the envelope). Further computing with algebraic packages (heavy algebraic machinery is sometimes needed, e.g. in order to work with polynomials, enabling to apply algorithms from the theory of Gröbner bases) may yield an implicit equation from the envelope, but not always.

Then back to the DGS an accurate plot of the envelope is obtained, which may confirm the conjecture established in first part.

We illustrate this kind of exploration with examples in 2D having the following characteristics:

1. The dynamical plot with DGS gives a good intuitive plot (a family of circles centered on an ellipse). Figure 1 shows two examples of envelopes of families of circles centered on an ellipse; both have two disjoint components.
2. The choice of two different dynamical tools provide different plots, one of them much more useful to establish a conjecture. Figure 1 has been obtained with GeoGebra's **Trace On**. The usage of a slider bar provides a kore uniform repartition of the circles.
3. An envelope is often thought as a "wrap" for the given family. The example in Figure 2 (a family of circles centered on an astroid) provides a

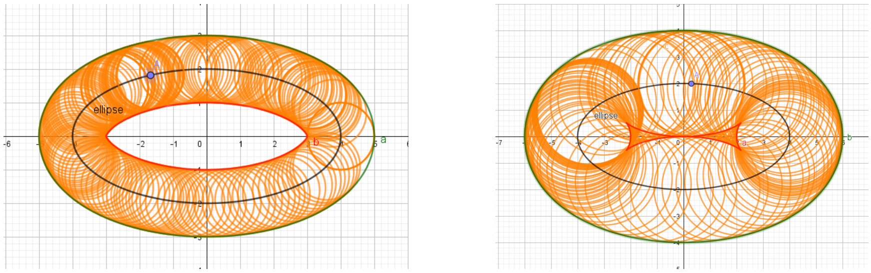

Fig. 1. Two envelopes of circles centered on an ellipse.

non-intuitive result: the envelope computed by analytic means lies "inside", and does not wrap the family of circles. The curve which seems to wrap the family (here an offset of the astroid) is harder to determine.

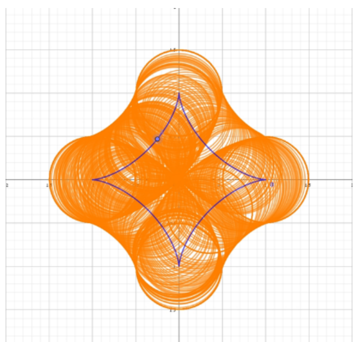

Fig. 2. Envelope and offset.

When computing an analytic presentation for the envelope, the CAS provides often a description of the envelope as the union of (disjoint) components. The examples emphasize the respective roles of these components. The dialog between the DGS and the CAS is revealed as an efficient tool for the automated exploration of envelopes. The CAS Giac has already been implemented into GeoGebra, further steps will increase efficiency of the dialog between the two kinds of software.

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Teaching of Geometry with GeoGebra software in students of the Primary Education Degree

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From our teaching experience in Primary School teacher training, it is very common that the future teachers have got numerous difficulties about geometrical concepts. Most of the students are not capable to visualize representations in the plane, much less in space. Possibly these difficulties are a consequence of a training based on memoristic learning of the different definitions, properties and formulas, without experimenting with any didactic resource for the acquisition of significant learning.

In order to solve this situation, we introduce the GeoGebra software in our teaching as a tool that could help to improve capabilities for reasoning and verifications, mathematical communication, problem-solving and understanding of concepts in Geometry.

This contribution present different classroom experiences about the didactic use of GeoGebra for teaching and learning Geometry in students of the Primary Education Degree. The objectives of the experience are, on the one hand, to encourage the use of GeoGebra, to improve the acquisition of geometric concepts and, at the same time, acquire knowledge about the difficulties and mistakes they make in their learning and in the use of this didactic tool; and on the other, encourage modes of action that will be useful in their professional life as teachers in *the digital era*.

A constructive educational approach to conics and quadrics allowed by the arrival of 3D-DGS

Eugenio Roanes-Lozano

Universidad Complutense de Madrid, Spain

Conics and quadrics are usually treated at Mathematics, Sciences and Engineering Schools. The availability of 3D Dynamic Geometry Systems (3D-DGS) with algebraic capabilites allows to introduce both of them in an attractive way. The different conics can be obtained (and viewed) as sections of a right circular cone ("Apollonius cone"). Allocating conveniently the cone allows a 3D-DGS to directly obtain the implicit equations of the conics [1]. Moreover, both degenerate and non-degenerate quadrics of revolution can be obtained (and plotted) in a constructive way too, as usually done for conics in 2D-DGS. For instance, an ellipsoid of revolution is the locus of points such that the sum of distances to the two foci is constant. Again, the implicit equations of the quadrics of revolution can be directly obtained by the 3D-DGS [2].

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TG4 Panel: Virtual reality, artificial intelligence and machine learning for mathematics education

Steven Van Vaerenbergh[†] and Theodosia Prodromou[‡]

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This is a summary of the *TG4 Panel: Virtual reality, artificial intelligence and machine learning for mathematics education* that took place at the Symposium on Artificial Intelligence for Mathematics Education (AI4ME), held at CIEM Castro Urdiales, February 28th - March 1st, 2020. Full list of authors: Theodosia Prodromou, Adrián Pérez, Martha Ivón Cárdenas, Roman Hašek, Steven Van Vaerenbergh, Alvaro Martínez Sevilla, and José Luis Rodriguez Blancas.

Summary

The *TG4 Panel: Virtual reality, artificial intelligence and machine learning for mathematics education* focused on artificial intelligence based technologies and their direct application to the learning and teaching of mathematics.

The session started with the invited talk by Theodosia Prodromou, "Orientations of research when integrating digital technologies in mathematics education: Emerging technologies and emerging types of learning" and was followed by four talks about AI in mathematics education:

1. Adrián Pérez: *Opportunities of machine learning in mathematical education*
2. Martha Ivón Cárdenas: *From graphs to neural networks: complexity and simplicity in the framework of mathematics*
3. Roman Hašek: *Artificial Intelligence, a promising agent of mathematical education*
4. Steven Van Vaerenbergh: *Recent advances in machine learning for mathematical reasoning*

Each talk touched on a different aspect of AI in mathematics education. Nonetheless, as we will conclude below, they allowed us to piece together the parts of a bigger picture of the interaction between students, teachers and AI technology.

In his talk, Adrián Pérez provided an overview of several areas in which machine learning (ML) is used for the analysis of educational data. A first such application is the analysis and prediction of student performance, which has been studied using regression techniques. As mentioned by Adrián, an interesting aspect of these techniques is that they allow to identify student features that are correlated, such as study time and performance. Next, a promising application of AI-based computer vision is sentiment classification, which, though currently in an experimental stage, could be used in real time in the classroom to provide the teacher with feedback. Lastly, two nascent research areas of ML applied to educational data are fair machine learning, which aims to avoid biases that may be underlying the data, and causal machine learning, which allows to estimate causal effects in student data that contain known causal directions.

Martha Ivón Cárdenas talked about strategies to teach secondary and high school students some of the fundamental concepts in AI. First, she showed how relational graphs can be taught with simple and intuitive examples, for instance by turning geometric constructions into graphs or by looking at real-world examples of graphs. Then, she described a bottom-up strategy to introduce the topic of artificial neural networks, starting at the mathematical equations that govern the functioning of individual neurons.

Roman Hašek explored the question “Can AI be applied as an agent to support the school education of pupils with specific needs?”, which concerns pupils with a disability or learning disorder as well as gifted pupils. The research towards this goal requires a specific focus on the collaboration between the teacher and the AI agent, such that the contributions of both are complimentary. Currently, a realistic scenario consists in employing the AI to collect data on the educational progress of the pupil, which is shared with the teacher and analyzed to suggest an individualized learning path.

Steven Van Vaerenbergh discussed recent developments in machine learning for automating mathematical reasoning, which is a key component in constructing systems that can model mathematical learners. Sparked by its success in other computational problems, machine learning is being applied in several areas that work towards automating mathematical reasoning: In the field of automated theorem proving, ML techniques are being applied to encode human provers’ intuitions and to predict the best next step in the demonstration of a theorem. Advances in neural networks for natural language processing are being used to train machines to solve word problems and to perform symbolic reasoning, yielding currently some limited but promising results. Finally, in the AI and ML communities there is a growing interest in automating abstract reasoning. The research in this area currently focuses on automating visual IQ tests, such as variants of Raven’s Progressive Matrices.

In the second part of the session, two practical projects were presented and demonstrated live:

5. Álvaro Martínez-Sevilla: *MonuMAI: Artificial Intelligence and Mathematics working over monuments*
6. José Luis Rodríguez Blancas: *Exploring dynamic geometry through immersive virtual reality with Neotrie VR*

Álvaro Martínez-Sevilla presented the project MonuMAI, which applies machine learning to recognize artistic styles and geometrical models in monumental architecture. He demonstrated the MonuMAI app, which is capable of extracting such mathematical knowledge from pictures taken by the user.

In order to obtain sufficient geometrical examples to train the deep learning model behind MonuMAI, Álvaro explained the novel technique of generating a data set of geometrical models through GeoGebra.

In the final talk, José Luis Rodríguez Blancas presented Neotrie VR, an immersive virtual reality environment for learning geometry. He demonstrated its practical uses in the classroom, by creating and manipulating 3D geometrical objects in a virtual world, which the user accesses through a virtual reality headset.

Conclusions

Students and teachers are at the center of the many AI-based innovations in mathematics education. Firstly, AI is being used to extract mathematical knowledge from the real world, facilitating its interpretation by the student. Virtual worlds are also being built, allowing students to interact directly with geometrical objects that may otherwise seem too abstract. The interactions of students with technology generate large amounts of data, out of which AI can extract concrete knowledge. As students become more and more used to being surrounded by AI-based technologies, it is also important to teach them some of the basic mathematical principles on which AI is based. Finally, in order to assist students in their learning process, research is being conducted in AI techniques that guide the student, whose long-term goal is to produce a machine that can model the mathematical learner.

Orientations of research when integrating digital technologies in mathematics education: Emerging technologies and emerging types of learning

Theodosia Prodromou

University of New England, Australia

Bringing the potential of enormous impact to human future life and bearing the name of an emerging technology, artificial intelligence is coming back strongly to our territory of research. I will discuss the orientations of research when integrating digital technologies in mathematics education, emerging technologies and emerging types of learning and how these orientations of research co-evolve and change education, rapidly and effectively.

Opportunities of Machine Learning in Mathematical Education

Adrián Pérez-Suay

Universitat de València, Spain

This talk is about the application of Machine Learning (ML) techniques in Mathematics Education data. In particular, I will cover some of the most relevant scenarios of ML like supervised learning, unsupervised learning, fair learning and dependence estimation together with causal inference. Other relevant aspect covered are the different sources of information related to students like databases or educational reports which could be useful to both detect and relate variables affecting students.

From graphs to neural networks: complexity and simplicity in the framework of mathematics

Martha-Ivón Cárdenas

Universitat Politècnica de Catalunya, Spain

In the computer sciences, many mathematical models are obtained to the aim of achieving automatic or machine-made processes that simulate human tasks. But human thinking is very complex, so are the models of Artificial Intelligence (AI). Calculating is very easy for a machine, but recommending an alternative process among several is not. In that sense, the graphs provided an adequate language to visualize those complex AI algorithms. Their great versatility has made them applicable to systems, networks, designs and predictions, being a beautiful example of neural networks at the service of mathematics and a powerful tool to address the organization of complex systems. From metro maps to distributions, from pattern recognition to creating social networks, and from airplane itineraries to image processing, all are great examples of mathematical modeling through graphs.

Artificial Intelligence, a promising agent of mathematical education

Roman Hašek

University of South Bohemia, Czech Republic

"As more and more of a student's education is experienced through a computer, data on their educational progress can be collected, leading to more personalized learning plans while assisting the teacher in identifying problem areas for students." – Loeffler, 2018 [1].

Inspired by the Loeffler's claim I am interested in the feasibility of a project focused on the application of AI to support and streamline the school education of pupils with specific needs, from the pupils with minor brain disfunction to the gifted pupils, to focus on their individual skills and demands.

Theoretical bases of AI implementation in mathematics education are stated in [2]. I would like to find out whether all building blocks of this implementation, whether of a technical, software or didactic nature, are sufficiently mature for actual use. If so, I offer to collaborate with any other interested parties in the development and mainly the testing of such solutions.

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Recent advances in machine learning for mathematical reasoning

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In the AI community, the problem of giving a machine the capability of understanding students' understanding, and hence, students' reasoning, is known as *student modeling* [1]. This problem has been studied for decades, and even early on, a myriad of different systems had been proposed [2]. Student modeling represents an inherently complex problem, related to the ambitious goal of artificial general intelligence, to which it has several parallels.

This talk focusses on techniques to automate mathematical reasoning, which represents an important component of student modeling systems. In particular, we discuss the application of machine learning (ML) techniques, which have recently achieved several breakthroughs in the automated resolution of complex tasks. We review several papers that advance the state of the art in ML for mathematical reasoning, and we draw connections to automated reasoning and more fundamental research performed in the field of abstract reasoning [3–9].

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MonuMAI: Artificial Intelligence and Mathematics working over monuments

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Mathematics explained about elements of everyday life or art have been a source of inspiration and educational projects in the last decade. The interest and use is growing in this field when in particular we approach the monumental heritage for its mathematical content, motivation and symbolic value. In this line we have developed our Mathematical Walks project, of which material and reference can be found at `https://paseosmatematicos.fundaciondescubre.es`.

Continuing along this line we have incorporated the tools of Artificial Intelligence in this mathematical analysis of monuments, through the techniques of Deep Learning, developing the MonuMAI app as a meeting platform for these three aspects for education or mathematical and historical-artistic dissemination[1].

In MonuMAI we have applied machine learning techniques for the recognition of artistic styles in monumental architecture through training based on their defining elements. But our proposal goes further. The two directions in which we currently work at MonuMAI intertwine mathematics and A.I. more deeply. On the one hand we are implementing computer vision techniques to automatically obtain a geometric model of each monument, with which to be able to make more detailed mathematical analysis, and on the other we are improving the learning of artistic elements through geometric models for training conducted with GeoGebra. This last process can be generalizable to other fields in which Deep Learning techniques are applied.

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Exploring dynamic geometry through immersive virtual reality with Neotrie VR

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Neotrie VR[1] is a software package that enables students to create, manipulate and play with 3D objects in a virtual reality scenario. It implements tools that are used naturally as if we were in real life, but also overcoming physical barriers. It also allows you to design and create VR scenes with activities for the math classroom, directly in the virtual environment.

[1] `http://www2.ual.es/neotrie/`

Author index

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The digital revolution that we have experienced since the last quarter of the twentieth century has had some influence, yet to be analysed and extended, on the way mathematics is made, taught and learned. While the rate of innovation in these technologies is growing exponentially, the potential impact of most information technologies on mathematical education remains to be fully exploited. In particular, several authoritative voices point out that the technology that will most likely transform education in the coming years is artificial intelligence (AI). Interestingly, today AI is mainly associated with technologies to automate tasks and lower costs, thus serving primarily the interests of the political-administrative, industrial and commercial world. In this scenario, the world of education and, more specifically, didactics, appears at best as a mere user of AI techniques developed in other fields, forgetting that AI should play a much more relevant role here, serving the human being who is doing his work as a mathematician or who is learning mathematics.

The AI4ME symposium at the International Centre for Mathematical Meetings (CIEM) in Castro Urdiales is a space for research and reflection to better understand the interconnected challenges of instrumental learning of mathematics and instrumental mathematics, taking advantage of the achievements and opportunities of Artificial Intelligence for Mathematical Education. This book of abstracts gathers the summaries of the talks presented at the symposium, as well as the conclusions of each of the four thematic groups.

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